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Heat Transfer in a Viscoelastic Fluid over a Stretching Sheet

M. S. Sarma

Applied Mathematics Division

and

B. Nageswara Rao

Structural Engineering Group, Vikram Sarabhai Space Centre, Trivandrum 695 022, India

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This paper presents an analytical solution for the heat transfer in the steady laminar flow of an incompressible viscoelastic fluid past a semi-infinite stretching sheet with power-law surface temperature or power-law surface heat flux, including the effects of viscous dissipation and internal heat generation or absorption. Important physical quantities such as the coefficients of skin friction and heat transfer are obtained. Asymptotic results for the temperature function for large Prandtl numbers are presented. © 1998 Academic Press

1. INTRODUCTION

The flow of an incompressible fluid past a moving surface has several engineering applications. The aerodynamic extrusion of plastic sheets, the cooling of a large metallic plate in a cooling bath, the boundary layer along material handling conveyers, the boundary layer along a liquid film in condensation processes and a polymer sheet or filament extruded continuously from a die, or a long thread traveling between a feel roll and a wind-up roll are examples of practical applications of a continuous flat surface. Non-Newtonian fluids have gained considerable importance because the power required in stretching a sheet in a viscoelastic fluid is less

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than when it is placed in a Newtonian fluid; and the heat transfer for a viscoelastic fluid is found to be less than that of Newtonian fluid. In view of these applications, viscoelastic boundary layer flow along a stretching sheet has been the subject of a large number of publications [1–6].

Vajravelu and Rollins [6] have studied the flow of an incompressible second-order fluid due to stretching of a plane elastic surface in the approximation of boundary layer theory. They have examined the effects of viscous dissipation (or frictional heating) and internal heat generation or absorption in a viscoelastic boundary layer flow. However, the energy equation of ref. 6 does not contain the terms of the work due to deformation. The exclusion of these terms from the energy equation is not in conformity with the inclusion of the viscous dissipation. Moreover, their expression for the dimensionless wall shear stress is erroneous. Dilute polymer solutions like 0.83% ammonium alginate in water and 5.4% polyisobutylene in cetane have approximate Prandtl numbers of 440 and 3, respectively. Ref. 6 gives the results for Prandtl numbers below 5. Apart from moderate Prandtl numbers, the solution for large Prandtl numbers appears to be of industrial importance. Hence the solutions for the temperature, the heat transfer characteristics, and their asymptotic limit for large Prandtl numbers are essential. The purpose of the present study is to reexamine the solution of the problem while including both the viscous dissipation and the work due to deformation.

2. FLOW ANALYSIS

The boundary layer equations for the heat transfer in the steady laminar flow of an incompressible viscoelastic fluid past a semi-infinite stretching sheet are [6]

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - \lambda\left\{\frac{\partial}{\partial x}\left(u\frac{\partial^2 u}{\partial y^2}\right) + \frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y^2} + v\frac{\partial^3 u}{\partial y^3}\right\} \tag{2}$$

$$\rho C_{\mathrm{p}}\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = \kappa\frac{\partial^2 T}{\partial y^2} + \mu\left(\frac{\partial u}{\partial y}\right)^2 - \rho\lambda\frac{\partial u}{\partial y}\frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) + Q(T - T_\infty). \tag{3}$$

The boundary conditions for the velocity field are

$$u = u_w = Bx; \quad v = 0 \qquad \text{at } y = 0,\ B > 0 \tag{4}$$

$$u \to 0 \qquad \text{as } y \to \infty. \tag{5}$$

Here u and v are the velocity components along the flow direction (x direction) and normal to the flow direction (y direction). The fluid is at rest, and the motion is created by the stretching of the sheet with velocity, u_w, which varies linearly with the distance from a fixed point on the sheet. Q is the specific heat generation rate. T is the temperature. T_∞ is a constant temperature of ambient fluid. λ is a positive parameter associated with the viscoelastic fluid. κ is the thermal conductivity. ρ is the density. $\nu = \mu/\rho$ is the kinematic viscosity. C_p is the specific heat at constant pressure. The third term on the right-hand side of Eq. (3) represents the work due to deformation that has been excluded from ref. 6.

The thermal boundary conditions depend on the type of heating process under consideration. The heat transfer analysis has been carried out for two different heating processes, namely (i) prescribed surface temperature (PST case) and (ii) prescribed surface heat flux (PHF case).

The thermal boundary conditions for the equation of energy (3) are

$$\text{PST case:} \quad T = T_w\left(\equiv T_\infty + A(x/L)^2\right) \qquad \text{at } y = 0 \tag{6}$$

$$\text{PHF case:} \quad -\kappa\frac{\partial T}{\partial y} = q_w = D(x/L)^2 \qquad \text{at } y = 0 \tag{7}$$

$$T \to T_\infty \qquad \text{as } y \to \infty. \tag{8}$$

Here A and D are constants. L is the characteristic length. q_w is the wall heat flux. The boundary conditions (6) and (7) indicate the heat transfer in a viscoelastic fluid over a stretching sheet with power-law surface temperature and power-law surface heat flux.

Introducing the stream function ψ as $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$, defining $\bar{\psi} = \psi/u_0 L = (\bar{x}/\sqrt{\mathrm{Re}})f(\eta)$, and $T = T_\infty + T_{sp}\bar{x}^2\theta(\eta)$, Eqs. (1)–(8) are transformed to

$$f'^2 - ff'' = f''' - \lambda_1\left(2f'f''' - f''^2 - ff''''\right) \tag{9}$$

$$\frac{1}{\mathrm{Pr}}\theta'' - f\theta' - (2f' - \alpha)\theta + \mathrm{Ec}\left(f''^2 - \lambda_1 f''(f'f'' - ff''')\right) = 0. \tag{10}$$

The boundary conditions for the momentum equation (9) are

$$f = 0, \quad f' = 1 \qquad \text{at } \eta = 0 \tag{11}$$

$$f' \to 0 \qquad \text{as } \eta \to \infty. \tag{12}$$

The thermal boundary conditions for the energy equation (10) are

$$\text{PST case:} \quad T_{sp} = A \Rightarrow \theta = 1 \qquad \text{at } \eta = 0 \tag{13}$$

$$\text{PHT case:} \quad T_{sp} = \frac{DL}{\kappa\sqrt{\mathrm{Re}}} \Rightarrow \theta' = -1 \qquad \text{at } \eta = 0 \tag{14}$$

$$\theta \to 0 \qquad \text{as } \eta \to \infty. \tag{15}$$

Here $\bar{x} = x/L$; $\bar{y} = y/L$; $u_0 = BL$; $\lambda_1 = (\lambda B)/\nu$; $\eta = \sqrt{\mathrm{Re}}\,\bar{y}$; $\mathrm{Re} = Lu_0/\nu$ is the Reynolds number; $\mathrm{Ec} = u_0^2/C_p T_{sp}$ is the Eckert number; $T_{sp} = A$ for PST case and $T_{sp} = DL/\kappa\sqrt{\mathrm{Re}}$ for the PHF case; $\mathrm{Pr} = \mu C_p/\kappa$ is the Prandtl number; $\alpha = Q/B\rho C_p$ is the heat source/sink parameter; L and u_0 are characteristic length and velocity, respectively; and primes denote differentiation with respect to η.

The transformation coordinate normal to the surface η indicates that the order of the momentum boundary layer thickness is $1/\sqrt{\mathrm{Re}}$. The skin friction coefficient C_f can be expressed as

$$C_f = \frac{\tau_{xy}|_{y=0}}{(1/2)\,\rho u_w^2} = \frac{2}{\sqrt{\mathrm{Re}\,\bar{x}}}(1 - 3\lambda_1) f''(0), \tag{16}$$

where τ_{xy} is the shear stress.

The local heat transfer coefficient, Nu (Nusselt number), for the case of prescribed surface temperature can be written as

$$\mathrm{Nu} = \frac{-\kappa(\partial T/\partial y)|_{y=0}}{\kappa(T_w - T_\infty)} x = -\sqrt{\mathrm{Re}\,\bar{x}}\,\theta'(0). \tag{17}$$

The momentum equation (9) is uncoupled from the energy equation (10). The solution for the momentum equation (9) with the boundary conditions (11) and (12) is

$$f(\eta) = \frac{1}{m}(1 - e^{-m\eta}) \qquad \text{for } 0 \le \lambda_1 < 1, \tag{18}$$

where $m = 1/\sqrt{1 - \lambda_1}$.

The solution for the energy equation (10) with the boundary conditions (13) and (15) for the case of prescribed surface temperature (PST case) is

$$\theta = (1 + q)\left(\frac{\xi}{-r}\right)^{p} \frac{M(p-2,\ s+1,\ \xi)}{M(p-2,\ s+1,\ -r)} - q\left(\frac{\xi}{-r}\right)^{2}, \tag{19}$$

where $\xi = -re^{-m\eta}$; $p = (r + s)/2$; $r = \mathrm{Pr}(1 - \lambda_1)$; $s = \sqrt{r(r - 4\alpha)}$; $q =$

$(\mathrm{Ec}r)/(4 - 2r + \alpha r)$. The Kummer's function,

$$M(a, b, z) = 1 + \sum_{n=1}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!},$$

satisfies the differential equation

$$z \frac{d^2 M}{dz^2} + (b - z) \frac{dM}{dz} - aM = 0,$$

and the Pochhammer symbol, $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$. For large values of z, $M(a, b, z) = (\Gamma(b)/\Gamma(a)) e^z z^{a-b}\{1 + O(1/z)\}$.

An examination of the solutions for θ given in Eq. (23) of ref. 6 and Eq. (19) reveals a change in the form of the parameter Ec, which turns out to be $\mathrm{Ec}(1 - \lambda_1)$, because of the presence of the terms of the work due to deformation in the energy equation.

The nondimensional surface velocity and temperature gradients obtained from Eqs. (18) and (19) are

$$f''(0) = -m \tag{20}$$

$$\theta'(0) = -rm(1+q)S_t + 2mq, \tag{21}$$

where

$$S_t = \frac{p}{r} - \left(\frac{p-2}{s+1}\right) \frac{M(p-1,\ s+2,\ -r)}{M(p-2,\ s+1,\ -r)}.$$

When $r \to 4/(2-\alpha)$, θ and $\theta'(0)$ in Eqs. (19) and (21) tend to

$$\theta \to \left(\frac{\xi}{-r}\right)^2 \left(1 - \frac{\mathrm{Ec}r}{s}\left\{\ln\left(\frac{\xi}{-r}\right) + \frac{1}{1+s}\int_{-r}^{\xi} M(1, 2+s, t)\, dt\right)\right. \tag{22}$$

$$\theta'(0) \to -rm\left(\frac{2}{r} - \frac{\mathrm{Ec}}{s}\left\{1 - \frac{r}{1+s} M(1,\ 2+s,\ -r)\right\}\right). \tag{23}$$

The solution for the energy equation (10) with the boundary conditions (14) and (15) for the case of prescribed surface heat flux (PHF case) is

$$\theta = \frac{1}{rS_t}\left(\frac{1}{m} + 2q\right)\left(\frac{\xi}{-r}\right)^p \frac{M(p-2,\ s+1,\ \xi)}{M(p-2,\ s+1,\ -r)} - q\left(\frac{\xi}{-r}\right)^2. \tag{24}$$

When $r \to 4/(2-\alpha)$, θ in Eq. (24) tends to

$$\theta \to \left(\frac{\xi}{-r}\right)^2 \left(\frac{1}{2m} - \frac{\mathrm{Ec}\,r}{s}\left\{\ln\left(\frac{\xi}{-r}\right) + \frac{1}{1+s}\int_{-r}^{\xi} M(1,\,2+s,\,t)\,dt\right)\right) \tag{25}$$

The asymptotic solution for $\theta(\eta)$ for large Prandtl numbers is obtained as follows. Defining $\theta = -qe^{-2m\eta} + \Theta$, the boundary layer equation for energy (10) and the corresponding boundary conditions (13)–(15) are transformed to

$$\frac{1}{\Pr}\Theta'' + f\Theta' + (\alpha - 2f')\Theta = 0 \tag{26}$$

$$\text{PST case:}\quad \Theta = 1 + q \qquad \text{at } \eta = 0 \tag{27}$$

$$\text{PHF case:}\quad \Theta' = -(1 + 2mq) \qquad \text{at } \eta = 0 \tag{28}$$

$$\Theta \to 0 \qquad \text{as } \eta \to \infty. \tag{29}$$

Since the thermal boundary layer thickness is on the order of $1/\sqrt{\Pr \mathrm{Re}}$, the transformation coordinate η in Eqs. (26)–(29) is further modified to $\tau = \sqrt{\Pr}\,\eta$. The stream function $f(\eta)$ and its derivative become $\sqrt{\Pr}f(\eta) \simeq \tau$, $f'(\eta) \simeq 1$. Then Eqs. (26)–(29) become

$$\ddot{\Theta} + \tau\dot{\Theta} + (\alpha - 2)\Theta = 0 \tag{30}$$

$$\text{PST case:}\quad \Theta = 1 + q \qquad \text{at } \tau = 0 \tag{31}$$

$$\text{PHF case:}\quad \dot{\Theta} = -\frac{1}{\sqrt{\Pr}}(1 + 2mq) \qquad \text{at } \tau = 0 \tag{32}$$

$$\Theta \to 0 \qquad \text{as } \tau \to \infty. \tag{33}$$

Here dots denote differentiation with respect to τ. The above approximations for the velocity field are valid only inside the thermal boundary layer.

The asymptotic solution of θ for the PST case obtained from Eqs. (30), (31), and (33) is

$$\theta = -qe^{-2m\eta} + (1 + q)\theta_{st}, \tag{34}$$

where

$$\theta_{st} = e^{-\tau^2/2}\left(M\left(\frac{3-\alpha}{2}, \frac{1}{2}, \frac{\tau^2}{2}\right) - S_{st}\tau M\left(\frac{4-\alpha}{2}, \frac{3}{2}, \frac{\tau^2}{2}\right)\right);$$

$$S_{st} = \sqrt{2}\,\frac{\Gamma(a_2)}{\Gamma(a_1)}; \quad a_1 = \frac{3-\alpha}{2}; \quad \text{and} \quad a_2 = \frac{4-\alpha}{2}.$$

The nondimensional surface temperature gradient $\theta'(0)$ obtained from Eq. (34) is

$$\theta'(0) = 2mq - (1+q)\sqrt{\Pr}\, S_{st}. \tag{35}$$

It is noticeable that the viscoelasticity has no effect on the heat transfer rate for $\mathrm{Ec} = 0$, according to the asymptotic formula (35).

The asymptotic solution of θ for the PHF case obtained from Eqs. (30), (32), and (33) is

$$\theta = -qe^{-2m\eta} + \frac{(1+2mq)}{\sqrt{\Pr}\, S_{st}}\theta_{st}. \tag{36}$$

3. RESULTS AND DISCUSSION

The problem of heat transfer in the steady laminar flow of an incompressible viscoelastic fluid over a stretching sheet with power-law surface temperature and power-law surface heat flux, including viscous dissipation and the work due to deformation and internal heat generation or absorption, has been examined. Closed-form solutions are obtained for the temperature, the heat transfer characteristics, and their asymptotic limit for large Prandtl numbers. It is noted from Eq. (20) that the magnitude of the nondimensional surface velocity gradient, $f''(0)$, increases with the non-Newtonian parameter, λ_1. The nondimensional surface temperature gradient, $\theta'(\eta = 0)$, for the case of prescribed surface temperature (PST case) and the surface temperature, $\theta(\eta = 0)$, for the case of prescribed surface heat flux are obtained. Table I gives the comparison of the nondimensional surface temperature gradient, $\theta'(0)$, for the PST case and

TABLE I
Comparison of the Nondimensional Surface Temperature Gradient, $\theta'(\eta = 0)$, for the PST Case and the Surface Temperature $\theta(\eta = 0)$ for the PHF Case

α	PST case: $\theta'(\eta = 0)$		PHF case: $\theta(\eta = 0)$	
	Ref. 6*	Eq. (21)	Ref. 6*	Eq. (24)
−0.1	−2.0820	−2.3388	0.67859	0.60228
0.0	−1.9737	−2.2361	0.70366	0.62380
0.1	−1.8602	−2.2187	0.73141	0.64757

$\Pr = 5$, $\lambda_1 = 0.2$, $\mathrm{Ec} = 1$.

*Excluding the terms of the work due to deformation in the energy equation.

TABLE II
Nondimensional Surface Temperature Gradient, $\theta'(0)$, for the PST Case and the Surface Temperature, $\theta(0)$, for the PHF Case

	PST case: $\theta'(\eta=0)$				PHF case: $\theta(\eta=0)$			
	Eq. (21)		Asymptotic result (35)		Eq. (24)		Asymptotic result (36)	
Pr	$\lambda_1 = 0.0$	$\lambda_1 = 0.2$	$\lambda_1 = 0.0$	$\lambda_1 = 0.2$	$\lambda_1 = 0.0$	$\lambda_1 = 0.2$	$\lambda_1 = 0.0$	$\lambda_1 = 0.2$
1	−1.000	−0.988	−1.394	−1.382	1.000	1.009	0.753	0.760
5	−2.219	−2.236	−2.261	−2.236	0.632	0.624	0.647	0.654
10	−3.049	−3.080	−3.142	−3.173	0.573	0.564	0.575	0.569
15	−3.664	−3.703	−3.769	−3.814	0.551	0.542	0.552	0.545
100	−8.716	−8.787	−8.836	−8.921	0.509	0.503	0.509	0.504
400	−16.76	−16.85	−16.88	−16.98	0.502	0.499	0.502	0.499

$Ec = 1$, $\alpha = 0$.

the surface temperature, $\theta(0)$, for the PHF case. The effect of the terms of the work due to deformation in the energy equation is appreciable. The results presented in Table II indicate that the magnitude of $\theta'(0)$ increases with the Prandtl number. Negligible variation of $\theta'(0)$ with respect to λ_1 is noticed for large values of Pr. An increasing Pr causes reduction in the thickness of the thermal boundary layer, and hence the asymptotic results of $\theta'(0)$ given by Eq. (35) tend to be closer to the exact values as Pr increases. The asymptotic formula (35) for the nondimensional surface temperature gradient can be directly utilized in practical applications for large Prandtl numbers.

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